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*(Continued from page 3 of cover.)*

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It is known that a network of electrically conducting elements, such as that indicated in Figure 1, with any two pairs of terminals *a*, *g* and *b*, *h*, behaves with respect to those terminals, at any single

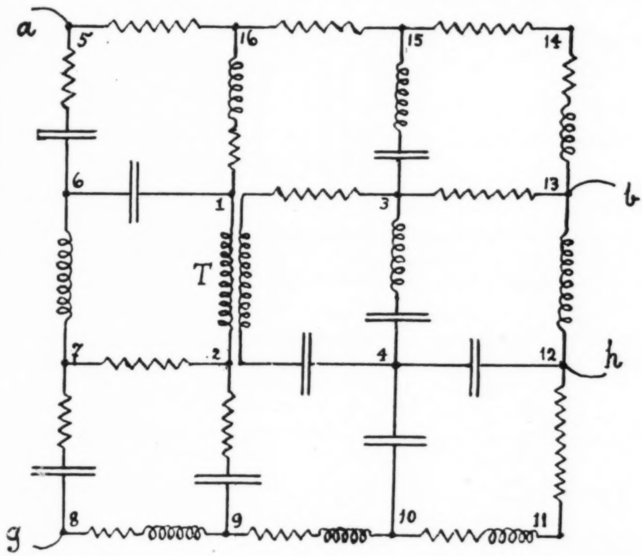



Figure 1. Example of a conducting network with two pairs of terminals *a*, *g* and *b*, *h*.

alternating-current frequency, like a certain $T$ as in Figure 2, or like a certain $\Pi$ as in Figure 3. If two of these terminals *b* and *h* are selected at one point, say No. 8 of the network, as a common terminal, the system becomes a "three-terminal network" and reduces to either a $T$ or a $\Delta$. A particular delta is indicated in Figure 4. In general it is dissymmetrical, or no two sides have the same impedance.

Let $\rho$ be the impedance between $a$ and $b$, $R_1$ that between $a$ and $g$, $R_2$ that between $b$ and $h$; all expressed in complex numbers of ohms; then we may denote their vector sum by $\Sigma$, or

$$\Sigma = \rho + R_1 + R_2 \qquad \text{ohms} \angle \quad (1)$$

and their planevector product $\Pi$ will be

$$\Pi = \rho\, R_1\, R_2 \qquad \text{ohms}^3 \angle \quad (2)$$

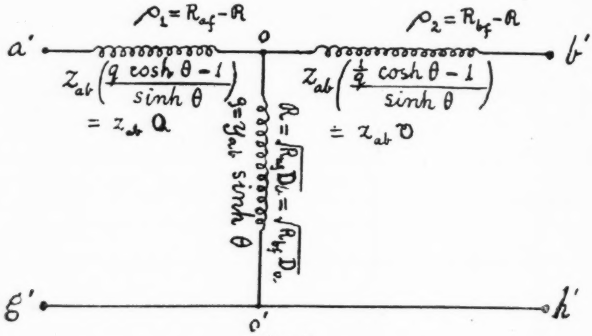



FIG. 2.

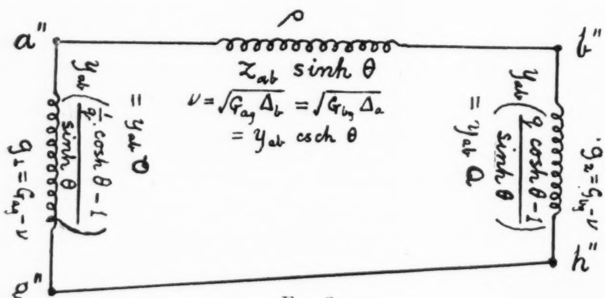



FIG. 3.

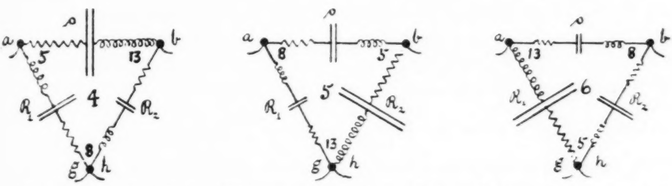
FIGURES 2 and 3. Dissymmetrical T and $\Pi$ corresponding to Figure 1.

*Constancy of $z_{ab}$ in the three different aspects of a three-terminal network.*

It is easily shown that the geomean surge impedance $z_{ab}$ of the system; i. e. the geometrical mean of the apparent surge impedance $z_{oa}$ at the $a$ $g$ terminals, and that $z_{ob}$ at the $b$ $h$ terminals, is:

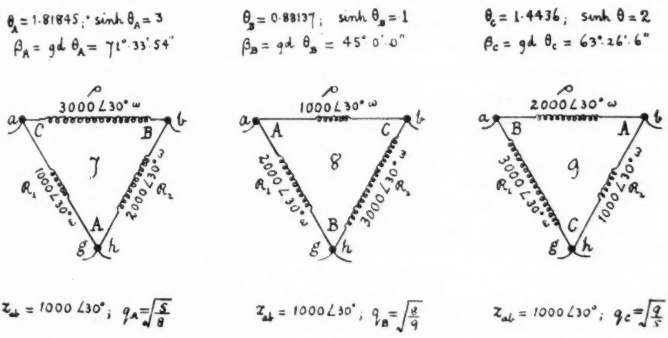
$$z_{ab} = \sqrt{z_{oa} \cdot z_{ob}} = \sqrt{\frac{\rho\, R_1\, R_2}{\rho + R_1 + R_2}} = \sqrt{\frac{\Pi}{\Sigma}} \qquad \text{ohms } \angle. \quad (3)$$

If then we shift the three terminals around the network of Figure 1; from the aspect shown in Figures 1 and 4 with $a$ on point 5, $b$ on 13, and $g$, $h$ on 8, or what may be called the 5–13 aspect, to the 8–5 aspect of Figure 5, with $a$ on point 8 and $b$ on 5, it is evident that the same triangle or delta of impedances will exist between the three terminals



FIGURES 4, 5 and 6. Three aspects of equivalent delta of a network at terminals $a$, $b$ and $g\,h$, for a given frequency.

$a$, $b$ and $gh$; but with $\rho$, $R_1$ and $R_2$ mutually interchanged. Their vector sum $\Sigma$, and their vector product $\Pi$, will, however, be the same as in Figure 4; so that the geomean surge impedance $z_{ab}$ of the network in the new aspect will, by (3), remain the same. Similarly, if the




FIGURES 7, 8 and 9. Three aspects of the equivalent delta of network.

terminals be again changed; so that $a$ is on point 13, $b$ on 8, and $gh$ on 5, the network will assume the 13–8 aspect of Figure 6; but its geomean surge impedance will again remain the same. Consequently, *in any*

*three-terminal network of conductors, with the three terminals disposed around three selected points in any of the three possible aspects, the geomean surge impedance of the network, or of its* T *and* Π, *with respect to the three terminals, will be the same.*

As an example, we may consider the simple case of an equivalent Π of a network, such as is shown in Figure 8. Here $AC = \rho = 1000 \angle 30°$ ohms, $AB = R_1 = 2000 \angle 30°$ ohms, $CB = R_2 = 3000 \angle 30°$ ohms. Hence $\Pi = 6 \times 10^9 \angle 90°$, and $\Sigma = 6 \times 10^3 \angle 30°$; so that $z_{ab} = \sqrt{\Pi/\Sigma} = 1000 \angle 30°$ ohms. Here we have the network in its $AC$ aspect. Figures 7 and 9 represent the same network in its $CB$, and $BA$ aspects, respectively. The geomean surge impedance $1000 \angle 30°$ ohms is the same, in all three cases.

*Values of sinh θ in the three aspects of a three-terminal network.*

It may be readily shown that the sine of the complex hyperbolic angle $\theta$ subtended by a network, with respect to three terminals, is

$$\sinh \theta = \frac{\rho}{z_{ab}} = \rho\sqrt{\frac{\Sigma}{\Pi}} \qquad \text{numeric } \angle. \quad (4)$$

so that in the three different aspects of Figures 4, 5 and 6, with $z_{ab}$ constant, $\sinh \theta$ is directly proportional to the architrave impedance $\rho$. In the cases of Figures 7, 8, and 9, $\sinh \theta$ is respectively 3.0, 1.0 and 2.0; from which the angle of the network in these three aspects is $\theta_A = 1.81845$, $\theta_B = 0.88137$ and $\theta_C = 1.4436$ hyperbolic radians, respectively.

*Consequently, in any three-terminal network, the sine of its angle θ in each of the three different aspects of the terminals, is directly proportional to the impedance between the terminals a, b.*

*Relations between the values of the inequality ratio q of a three-terminal network.*

The value of the inequality ratio $q$ for a dissymmetrical Π or Δ is known to be

$$q = \sqrt{\frac{\rho + R_2}{\rho + R_1} \times \frac{R_1}{R_2}} = \sqrt{\frac{g_2 + \nu}{g_1 + \nu}} \qquad \text{numeric } \angle \quad (5)$$

and this is in general different in each aspect of a three-terminal network, but if we denote the three different values of $q$ for these aspects by $q_A$, $q_B$ and $q_C$ respectively; then we find from (5) that

$$q_A \cdot q_B \cdot q_C = 1 \angle 0° \qquad \text{numeric} \quad (6)$$

In the case of Figures 7, 8, and 9, $q_A = \sqrt{\frac{5}{8}}$, $q_B = \sqrt{\frac{8}{9}}$, and $q_C = \sqrt{\frac{9}{5}}$.

*Consequently, the continued product of the three inequality ratios of a three-terminal network, in its three aspects, is always equal to unity.*

*Equality between the planevector sum and planevector product of the sines of the angles subtended by a three-terminal network in its three different aspects.*

We have already seen, by (4) that $\sinh\theta = \rho/z_{ab}$, in each aspect of the network. Consequently, if $\theta_A$, $\theta_B$ and $\theta_C$ are the three angles of the network in its three different aspects, we have:

$$\sinh\theta_A \cdot \sinh\theta_B \cdot \sinh\theta_C = \frac{\Pi}{z^3_{ab}} = \Pi \cdot \frac{\Sigma}{\Pi} \cdot \sqrt{\frac{\Sigma}{\Pi}} = \Sigma\sqrt{\frac{\Sigma}{\Pi}} \quad \text{numeric} \angle \ (7)$$

Moreover,

$$\sinh\theta_A + \sinh\theta_B + \sinh\theta_C = \frac{\Sigma}{z_{ab}} = \Sigma\sqrt{\frac{\Sigma}{\Pi}} \quad \text{“} \quad \angle \ (8)$$

Consequently, in any three-terminal network,

$$\sinh\theta_A + \sinh\theta_B + \sinh\theta_C = \sinh\theta_A \cdot \sinh\theta_B \cdot \sinh\theta_C \quad \text{“} \quad \angle \ (9)$$

or the sum of the sines of the three hyperbolic angles is equal to their product. Thus, in the simple case of Figures 7, 8, and 9, with $\sinh\theta_A = 3$, $\sinh\theta_B = 1$, and $\sinh\theta_C = 2$, both the sum and product are equal to 6.

*Relations between the gudermannian angles of a three-terminal network represented by a Π of equislope impedances.*

Formula (9) connecting the sines of the three hyperbolic angles of a three-terminal network, in its three successive aspects, is of general application, whatever the impedances in the network may be. In the more limited case, however, when the three impedances of the equivalent delta of the network, at the three terminals, have the same slope (same powerfactor and same reactance factor), the angles $\theta_A$, $\theta_B$ and $\theta_C$ will have no imaginary components, or will be real hyperbolic angles. Under these conditions, they are connected by an additional relation. It is well known that $\sinh\theta$, the sine of any real

hyperbolic angle $\theta$, is numerically equal to the tangent of a related circular angle, commonly called the "gudermannian angle." Thus

$$\sinh\theta = \tan\beta \qquad \text{numeric (10)}$$

where $$\beta = gd\,\theta \qquad \text{circular radians or degrees (11)}$$

Consequently, when $\theta_A$, $\theta_B$, and $\theta_C$ are all imaginaryless hyperbolic angles, we may substitute (10) in (9), and obtain

$$\tan\beta_A + \tan\beta_B + \tan\beta_C = \tan\beta_A \cdot \tan\beta_B \cdot \tan\beta_C \qquad \text{numeric (12)}$$

or the three circular gudermannian angles of the three-terminal network have the sum of their tangents equal to the product of their tangents. This condition of equality between the product and sum of three tangents in circular trigonometry, is a well known property of the interior angles of each and every plane triangle.

Consequently, *in any three-terminal network for which* $\theta_A$, $\theta_B$ and $\theta_C$ *are real, there exists a corresponding characteristic type of plane triangle, with interior angles* $\beta_A$, $\beta_B$ *and* $\beta_C$, *these interior angles being respectively the gudermannians of* $\theta_A$, $\theta_B$ *and* $\theta_C$.

Thus, in the simple case of Figures 7, 8 and 9, and in which $\theta_A$, $\theta_B$ and $\theta_C$ have no imaginary components, $\beta_B = 45°$, $\beta_C = 63°\,.26'\,.6''$ and $\beta_A = 71°\,.33'\,.54''$. The sum of these circular angles is $\pi$ radians, or 180°; i. e.

$$\beta_A + \beta_B + \beta_C = gd\,\theta_A + gd\,\theta_B + gd\,\theta_C = \pi \qquad \text{circular radians (13)}$$

The plane triangle corresponding to such a network is defined only by its internal angles $\beta_A$, $\beta_B$, $\beta_C$, and its sides are indeterminate. An infinite series of similar plane triangles have the same interior angles. The ratios of the sides $a$, $b$, $c$ of all these plane triangles are defined, however, by the relation:

$$a : b : c :: \sin\beta_A : \sin\beta_B : \sin\beta_C :: \tanh\theta_A : \tanh\theta_B : \tanh\theta_C \qquad (14)$$

The sides of the characteristic plane triangle corresponding to a three-terminal network, having real hyperbolic angles $\theta$, are thus respectively proportional to the tangents of the hyperbolic angles, or to the sines of their gudermannians.

For the particular network of Figures 7, 8 and 9, the plane triangle $A', B', C'$, of Figure 10 is of the characteristic type, the interior angles of this triangle are respectively the gudermannians of the network hyperbolic angles $\theta_A$, $\theta_B$ and $\theta_C$.

In order that the angles $\theta_A$, $\theta_B$ and $\theta_C$ of a three-terminal network

shall each and all be imaginaryless, it is necessary and sufficient that the three impedances of the delta in Figures 4, 5 and 6 shall all be either reactanceless (pure resistances) or have the same slope, as in

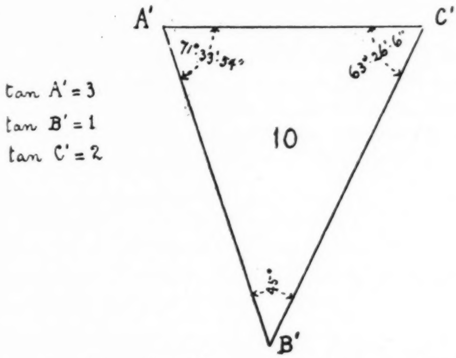
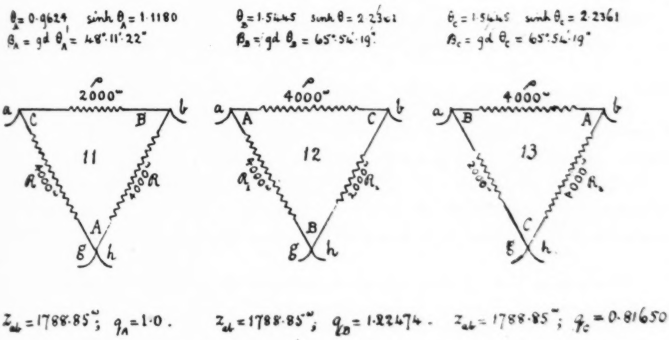



FIGURE 10. Plane triangle corresponding to network of Figures 7, 8 and 9.




FIGURES 11, 12 and 13. Equivalent delta of network, with two sides equal.

Figures 7, 8 and 9. Thus, let $\rho$ be the impedance $AC$, Figure 8, having a size $|\rho|$ and a slope $\overline{\rho}$. In Figure 8, $|\rho| = 1000$ and $\overline{\rho} = 30°$. Then if $R_1 = m\rho$ and $R_2 = n\rho$, $m$ and $n$ will be real numbers, if $\overline{R_1} = \overline{R_2} = \overline{\rho}$, or all three impedances have the same slope. Then, by (4),

$$\sinh \theta_B = \rho \sqrt{\frac{(m+n+1)\,\rho}{mn\,\rho^3}} = \sqrt{\frac{m+n+1}{mn}} \qquad \text{numeric}$$

This is a real number; because $m$ and $n$ are reals. The same reasoning applies to $\sinh\theta_A$ and $\sinh\theta_B$. Hence $\theta_A$, $\theta_B$ and $\theta_C$ are all real, if $\rho$, $R_1$, $R_2$ have the same slope.

*Particular case of two equal impedances in a Three-Terminal Network Delta.*

If two of the impedances representing a network at the terminals $a$, $b$ amd $gh$ are equal, as in the simple case of Figures 11, 12 and 13; then in the aspect (Figure 11) where the two pillars $R_1$ and $R_2$ are equal, suppose $\rho = m\,R_1 = m\,R_2 = m\,R$ ohms, or $q = 1$.

Then $$\Sigma = \rho\,\frac{(m+2)}{m} \qquad \text{ohms} \angle \quad (16)$$

and $$\Pi = \frac{\rho^3}{m^2} \qquad \text{ohms}^3 \angle \quad (17)$$

so that by (3) $$z_{ab} = z_o = \frac{\rho}{\sqrt{m(m+2)}} \qquad \text{ohms} \angle \quad (18)$$

and by (4) $$\sinh\theta = \sqrt{m(m+2)} \qquad \text{numeric} \angle \quad (19)$$

or $$\cosh\theta = m+1 \qquad \text{numeric} \angle \quad (20)$$

also $$\text{versh}\,\theta = \cosh\theta - 1 = m \qquad \text{numeric} \angle \quad (21)$$

Consequently, the angle $\theta$ of the network, in this aspect, will have $m$ for its versed sine. In general, $m$ will be complex.

In the case of Figures 11, 12 and 13, in which the delta sides are all reactanceless, or simple resistances, there is a corresponding gudermannian triangle and it is isoceles. Here $\cosh\theta_A = 1.5$, or $m = 0.5$.

*Particular case of three equal impedances in a Three-Terminal Network Delta.*

When the equivalent delta of a network, with respect to the terminals $a$, $b$, and $gh$ happens to contain three equal impedances; i. e. equal both as to size and slope; then $q = 1$, $m = 1$, $\Sigma = 3\rho$, and $\pi = \rho^3$.

$$z_{ab} = z_o = \frac{\rho}{\sqrt{3}} \qquad \text{ohms} \angle \quad (22)$$

$$\sinh\theta = \sqrt{3} \qquad \text{numeric} \quad (23)$$

$$\cosh \theta = 2 \qquad \text{numeric (24)}$$

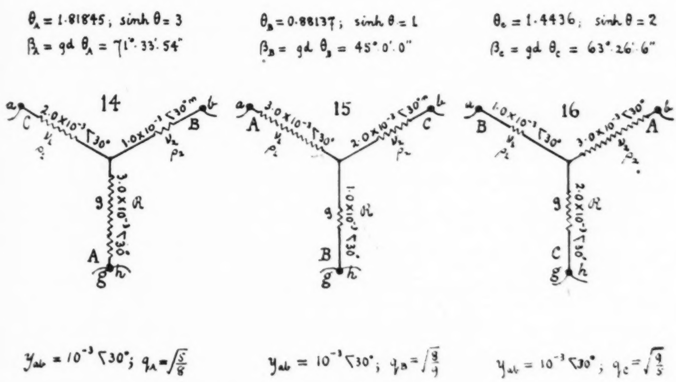
$$\text{versh}\, \theta = 1 \qquad \text{numeric (25)}$$

The three angles $\theta_A$, $\theta_B$ and $\theta_C$ will be equal, and each will have unit versed sine. This angle is 1.31696 $\angle 0°$ hyps. There is thus a corresponding gudermannian triangle and it is equilateral, each interior angle being 60°, or

$$gd\, \theta = \frac{\pi}{3} \qquad \text{circular radians (26)}$$

*Particulars concerning the Equivalent* T *of a Three-Terminal Network.*

We have hitherto confined attention to a delta connection of three impedances as representing a conducting network, at a given single frequency, with respect to three terminals $a$, $b$ and $gh$. We may,




FIGURES 14, 15 and 16. Equivalent star of network in three aspects.

however, replace such a delta of impedances by an equivalent star or $T$, as in Figures 14, 15 and 16, which correspond to Figures 7, 8 and 9. Then it will be found that if $\nu_1$, $\nu_2$ and $g$ are the three planevector admittances of the three star branches, and if

$$\Sigma' = \nu_1 + \nu_2 + g \qquad \text{mhos} \angle \text{ (27)}$$

$$\Pi' = \nu_1 \nu_2 g \qquad \text{mhos}^3 \angle \text{ (28)}$$

the geomean surge admittance $y_{ab} = 1/z_{ab}$ of the system, or of its equivalent $T$, is

$$y_{ab} = \sqrt{\frac{\Pi'}{\Sigma'}} \qquad \text{mhos} \angle \quad (29)$$

$$q = \sqrt{\frac{\rho_1 + R}{\rho_2 + R}} \qquad \text{numeric} \angle \quad (30)$$

and

$$\sinh \theta = \frac{g}{y_{ab}} = g\sqrt{\frac{\Sigma'}{\Pi'}} \qquad \text{numeric} \angle \quad (31)$$

It will be evident that the geomean surge admittance of any three-terminal network, presented as a dissymmetrical $T$, will be the same in all three aspects. Again, as in (6)

$$q_A \cdot q_B \cdot q_C = 1 \angle 0° \qquad \text{numeric} \quad (32)$$

Moreover, as in (9)

$$\sinh \theta_A + \sinh \theta_B + \sinh \theta_C = \sinh \theta_A \cdot \sinh \theta_B \cdot \sinh \theta_C \qquad \text{numeric} \angle \quad (33)$$

If all the sines are real, we find a corresponding characteristic gudermannian triangle, in accordance with (13).

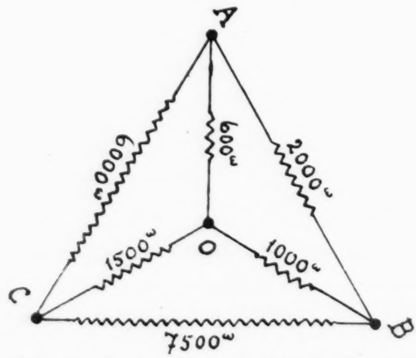



FIGURE 17. Six element network with four junction points.

Figure 17 shows a particular network of six elements $AO$, $BO$, $CO$, $AB$, $BC$, $CA$ all taken as reactanceless, or pure resistances. There are just 12 ways in which, using the four junctions $A$ $B$ $C$ and $O$ for

Table of Data Relating to the six-Element Resistance Network of Fig. 17

| | Triangle ABC | | | Triangle OBA | | | Triangle OAC | | | Triangle OBC | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Terminal Pairs | AC-BC | BA-CA | CB-AB | OA-BA | BO-AO | AB-OB | OC-AC | AO-CO | CA-OA | OC-BC | BO-CO | CB-OB |
| $R_{ag}$ | 666.66 | 750.0 | 1200 | 348.0 | 619.217 | 433.915 | 367.50 | 458.658 | 1046.26 | 466.66 | 655.223 | 1047.38 |
| $R_{af}$ | 1333.3̇ | 833.3̇ | 1500 | 468.33 | 668.3̇3 | 833.33 | 1068.3̇3 | 468.33 | 1333.33 | 1068.33 | 668.33 | 1500 |
| $R_{bg}$ | 750.0 | 1200 | 666.6̇ | 619.217 | 433.915 | 348.0 | 458.658 | 1046.27 | 367.50 | 665.226 | 1047.38 | 466.66 |
| $R_{bf}$ | 1500 | 1333.3 | 833.3̇ | 833.33 | 468.333 | 668.333 | 1333.33 | 1068.33 | 468.33 | 1500 | 1068.33 | 668.33 |
| $\tanh^2\theta = \frac{R_{ag}}{R_{af}} = \frac{R_{bg}}{R_{bf}}$ | 0.5 | 0.9 | 0.8 | 0.74306 | 0.92651 | 0.52070 | 0.34399 | 0.97934 | 0.78470 | 0.43682 | 0.98039 | 0.64920 |
| $\tanh\theta$ | 0.70711 | 0.94868 | 0.89443 | 0.86201 | 0.96255 | 0.72159 | 0.58651 | 0.98962 | 0.88583 | 0.66092 | 0.99014 | 0.83592 |
| $\theta$ | 0.88137 | 1.81845 | 1.44436 | 1.30112 | 1.97455 | 0.91097 | 0.67233 | 2.62776 | 1.4022 | 0.79445 | 2.65402 | 1.206[illegible] |
| $Z_a = \sqrt{R_{ag}\ R_{af}}$ | 942.809 | 730.569 | 1341.64 | 403.708 | 643.307 | 601.328 | 626.59 | 463.47 | 1181.11 | 706.085 | 661.746 | 1253.[illegible] |
| $Z_b = \sqrt{R_{bg}\ R_{bf}}$ | 1060.66 | 1264.91 | 745.356 | 718.341 | 450.796 | 482.265 | 782.01 | 1057.24 | 414.86 | 441.382 | 1057.81 | 558.[illegible] |
| $Z_{ab} = \sqrt{Z_{oa}\ Z_{ob}}$ | 1000 | 1000 | 1000 | 538.517 | 538.517 | 538.517 | 700.0 | 700.0 | 700.0 | 836.66 | 836.66 | 836.[illegible] |
| $y_{ab} = \sqrt{y_{oa}\ y_{ob}} \times 10^3$ | 1.0 | 1.0 | 1.0 | 1.85695 | 1.85695 | 1.85695 | 1.42857 | 1.42857 | 1.42857 | 1.19523 | 1.19523 | 1.195[illegible] |
| $q = \sqrt{Z_{oa} / Z_{ob}}$ | 0.94281 | 0.79057 | 1.34164 | 0.74967 | 1.10459 | 1.11664 | 0.89514 | 0.162093 | 1.68730 | 0.84393 | 0.70094 | 1.448[illegible] |
| | | $\pi = 1$ | | | $\pi = 1$ | | | $\pi = 1$ | | | $\pi = 1$ | |
| $\frac{1}{q} = \sqrt{Z_{ob} / Z_{oa}}$ | 1.06066 | 1.26493 | 0.74536 | 1.33393 | 0.83711 | 0.89555 | 1.11716 | 1.51036 | 0.59266 | 1.18493 | 1.26432 | 0.[illegible] |
| | | $\pi = 1$ | | | $\pi = 1$ | | | $\pi = 1$ | | | | |
| $Q = \frac{q \cosh\theta - 1}{\sinh\theta}$ | 0.333̇ | 0.50 | 1.0 | 0.28164 | 0.95943 | 0.58804 | 0.14524 | 0.52381 | 1.38095 | 0.14143 | 0.65735 | 1.33[illegible] |
| $U = \frac{\frac{1}{q}\cosh\theta - 1}{\sinh\theta}$ | 0.50 | 1.0 | 0.333̇ | 0.95943 | 0.58804 | 0.28164 | 0.52381 | 1.38095 | 0.14524 | 0.65735 | 1.33[illegible] | 0.14[illegible] |
| $\sinh\theta$ | 1.0 | 3.0 | 2.0 | 1.70058 | 3.55066 | 1.04229 | 0.72414 | 6.88525 | 1.90909 | 0.88[illegible] | 7.07076 | 1.52[illegible] |
| | | $\Sigma = \pi = 6.0$ | | | $\Sigma = \pi = 6.29353$ | | | $\Sigma = \pi = 9.5184$ | | | $\Sigma = \pi = $ [illegible] | |
| $\cosh\theta$ | 1.414 | 3.1623 | 2.2361 | 1.97281 | 3.68874 | 1.44444 | 1.23466 | 6.9575 | 2.15514 | 1.33253 | 7.14073 | 1.826[illegible] |
| $gd\ \theta$ | 45°0′0″ | 71°33′54″ | 63°26′6″ | 59°32′35″ | 74°16′15″ | 46°11′10″ | 35°54′35″ | 81°44′10″ | 62°21′15″ | 41°22′13″ | 81°56′59″ | 56°[illegible] |
| | | 180° | | | 180° | | | 180° | | | 180° | |
| Elements of Equivalent Π: $\rho$ | 1000 | 3000 | 2000 | 915.789 | 1912.09 | 561.29 | 506.846 | 4819.67 | 1336.37 | 736.842 | 5415.44 | 1272.7[illegible] |
| $\nu \times 10^3$ | 1.0 | 0.333̇ | 0.5 | 1.09195 | 0.52299 | 1.78161 | 1.97279 | 0.20748 | 0.74830 | 1.35714 | 0.184658 | 0.785[illegible] |
| $R_1$ | 2000 | 1000 | 3000 | 561.29 | 915.789 | 1912.09 | 1336.37 | 506.896 | 4819.67 | 1272.73 | 736.842 | 5415.44 |
| $R_2$ | 3000 | 2000 | 1000 | 1912.09 | 561.290 | 915.784 | 4819.67 | 1336.37 | 506.896 | 5415.44 | 1272.73 | 736.842 |
| Elements of Equivalent T: $\rho_1$ | 333.3̇ | 500 | 1000 | 151.66 | 516.66 | 316.66 | 101.66 | 366.66 | 966.66 | 118.33 | 550.0 | 950.00 |
| $\rho_2$ | 500 | 1000 | 333.3̇ | 516.66 | 316.66 | 151.66 | 366.66 | 966.66 | 101.66 | 550.0 | 950.0 | 118.33 |
| $R$ | 1000 | 333.3̇ | 500 | 316.66 | 151.66 | 516.66 | 966.66 | 101.66 | 366.66 | 950.0 | 118.33 | 550.0 |
| $g \times 10^3$ | 1.0 | 3.0 | 2.0 | 3.15789 | 6.59341 | 1.93548 | 1.03448 | 9.83607 | 2.72727 | 1.05263 | 8.45070 | 1.8[illegible] |

terminal points, this network can be connected to three terminals so as to form a three-terminal delta network; i. e. three aspects to each of the four triangles *ABC, AOB, AOC* and *BOC*. Each of these 12 deltas provides also an equivalent star network. They are all dissymmetrical. They are analyzed and presented in the accompanying Table. It will be seen that in each of the four triple-aspect groups, the following properties are found:—

(1) $z_{ab}$ and its reciprocal $y_{ab}$ are constant for each group.

(2) The triple $q$ product of each group is equal to unity.

(3) " " $1/q$ " " " " " " " "

(4) There are three values for $Q$ and $\delta$, distributed among each group.

(5) There is equality between the sums and products of $\sinh\theta$ in each group.

(6) The three gudermannian angles of each group sum to 180°, or there exists a corresponding plane triangle for each group.

*Summary of Conclusions, relating to a Three-Terminal Network, at a single alternating-current frequency (including zero frequency) in the steady state.*

(1) In the three aspects of the three terminals, the geomean surge impedance $z_{ab}$ and its reciprocal, the geomean surge admittance $y_{ab}$, are constant.

(2) In the three aspects of a delta network, $\sinh\theta$, the sine of the angle of the network is proportional to the impedance connecting the $a$, $b$ terminals.

(3) The product $q_A \cdot q_B \cdot q_C$ of the three inequality ratios is equal to unity.

(4) The planevector sum of the sines of the three angles of the network is equal to their planevector product.

(5) When the three hyperbolic angles $\theta_A$, $\theta_B$, $\theta_C$ are real, the three corresponding gudermannian circular angles make up a sum of $\pi$ radians or 180°. Consequently, these gudermannians define a triangle, or family of similar plane triangles.

(6) When two of the delta impedances of a network are equal, and form the pillars of the equivalent $\pi$, the versed sine of the angle of the network is $m$ where $m = \rho/R$.

(7) When all three of the delta impedances of a network are equal in size and in slope, the angles in the three aspects are equal, real, and have a versed sine of unity. The angle is 1.317 hyps, nearly, and the gudermannian triangle is equilateral.

(8) Similar relations affect the equivalent $T$ or star of a network.

*List of Symbols Employed.*

| | |
|---|---|
| $a, b, c$ | lengths of sides of a plane triangle. |
| $\beta$ | a circular angle, the gudermannian of a network hyperbolic angle (circular radians or degrees) |
| $D = R_f - R_g$ | difference in impedance due to shorting at the opposite end terminals (ohms $\angle$). |
| $\Delta = G_g - G_f$ | difference in admittance due to shorting at the opposite end terminals (mhos $\angle$). |
| $G_{ag}$ | admittance measured at terminals $a, g$, when the $bh$ pair are shorted (mhos $\angle$). |
| $G_{af}$ | admittance measured at terminals $a, g$, when the $b, h$, pair are freed (mhos $\angle$). |
| $G_{bg}$ | admittance measured at terminals $a, g$, when the $b, h$, pair are shorted (mhos $\angle$). |
| $G_{bf}$ | admittance measured at terminals $b, h$, when the $a, g$, pair are freed (mhos $\angle$). |
| $g$ | admittance of the staff leak in an equivalent $T$ (mhos $\angle$). |
| $g_1, g_2$ | admittance of the two pillar leaks of an equivalent $\pi$ (mhos $\angle$). |
| $gd\,\theta$ | gudermannian of a real hyperbolic angle $\theta$. (Circular radians). |
| $\theta$ | hyperbolic angle, real or complex (hyperbolic radians or hyps $\angle$). |
| $\theta_A, \theta_B, \theta_C,$ | hyperbolic angles of a network in three successive aspects (hyps $\angle$). |
| $m, n$ | real numbers, integral or fractional for equislope impedances, also $m$ a complex number for case of two equal pillar impedances. |
| $\nu$ | admittance of the architave of an equivalent $\Pi$ or delta (mhos $\angle$). |

$\nu_1, \nu_2$ admittance of the arms of an equivalent $T$ or star (mhos $\angle$).

$\Pi$ product of three impedances forming the sides of an equivalent delta (ohms$^3$ $\angle$).

$\Pi'$ product of three admittances forming the branches of an equivalent star (mhos$^3$ $\angle$).

$\Pi$ a delta connection of three impedances simulating the properties of a network at three terminals.

$\pi = 3.14159\ldots$

$Q = \dfrac{q \cosh \theta - 1}{\sinh \theta}$ a factor in the determination of an equivalent $T$ or $\Pi$ (numeric $\angle$).

$\mathfrak{Q} = \dfrac{\frac{1}{q} \cosh \theta - 1}{\sinh \theta}$ a factor in the determination of an equivalent $T$ or $\Pi$ (numeric $\angle$).

$q = \sqrt{z_{oa}/z_{ob}}$ inequality ratio of a system.

$q_A, q_B, q_C$ inequality ratios of a three-terminal system in its three successive aspects. (Numeric $\angle$).

$R$ impedance of a $T$ staff leak (ohms $\angle$).

$R_1, R_2$ impedances of the pillar leaks of a $\Pi$ (ohms $\angle$).

$R_{af}$ impedance of a network measured at terminals $a, g$, when freed at $b, h$, (ohms $\angle$).

$R_{ag}$ impedance of a network measured at terminals $a, g$, when shorted at $b, h$, (ohms $\angle$).

$R_{bf}$ impedance of a network at terminals $b, h$, when freed at $a, g$, (ohms $\angle$).

$R_{bg}$ impedance of a network measured at terminals $b, h$, shorted at $a, g$, (ohms $\angle$).

$\rho$ impedance of the architrave of a $\Pi$ (ohms $\angle$).

$\Sigma$ sum of three impedances forming the sides of a $\Pi$ or delta (ohms $\angle$).

$\Sigma'$ sum of three admittances forming the branches of a $T$ (mhos $\angle$).

$y_{ab} = 1/z_{ab}$ surge admittance of a system (mhos $\angle$).

$z_o$ surge impedance of a symmetrical system (ohms $\angle$).

$z_{oa}, z_{ob}$ surge impedances of a dissymmetrical system from the $a$, $g$, and $b$, $h$, terminals, respectively (ohms $\angle$).

$z_{ab} = \sqrt{z_{oa} \cdot z_{ob}}$ geomean surge impedance of a dissymmetrical system (ohms $\angle$).

$|z|$ size or modulus of a complex quantity $z$ (numeric).

$\underline{z}$ slope or argument of a complex quantity $z$ (radians or degrees.)

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